

TRACE INEQUALITIES ON A GENERALIZED WIGNER-YANASE SKEW INFORMATION

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ABSTRACT. We introduce a generalized Wigner-Yanase skew information and then derive the trace inequality related to the uncertainty relation. This inequality is a non-trivial generalization of the uncertainty relation derived by S. Luo for the quantum uncertainty quantity excluding the classical mixture. In addition, several trace inequalities on our generalized Wigner-Yanase skew information are argued.

1. INTRODUCTION

Wigner-Yanase skew information

$$(1.1) \quad \begin{aligned} I_\rho(H) &\equiv \frac{1}{2} \text{Tr} \left[\left(i [\rho^{1/2}, H] \right)^2 \right] \\ &= \text{Tr}[\rho H^2] - \text{Tr}[\rho^{1/2} H \rho^{1/2} H] \end{aligned}$$

was defined in [8]. This quantity can be considered as a kind of the degree for non-commutativity between a quantum state ρ and an observable H . Here we denote the commutator by $[X, Y] \equiv XY - YX$. This quantity was generalized by Dyson

$$\begin{aligned} I_{\rho, \alpha}(H) &\equiv \frac{1}{2} \text{Tr} \left[(i [\rho^\alpha, H]) (i [\rho^{1-\alpha}, H]) \right] \\ &= \text{Tr}[\rho H^2] - \text{Tr}[\rho^\alpha H \rho^{1-\alpha} H], \quad \alpha \in [0, 1] \end{aligned}$$

which is known as the Wigner-Yanase-Dyson skew information. It is famous that the convexity of $I_{\rho, \alpha}(H)$ with respect to ρ was successfully proven by E.H. Lieb in [5]. From the physical point of view, an observable H is generally considered to be an unbounded operator, however in the present paper, unless otherwise stated, we consider $H \in B(\mathcal{H})$, where $B(\mathcal{H})$ represents the set of all bounded linear operators on the Hilbert space \mathcal{H} , as a mathematical interest. We also denote the set of all self-adjoint operators (observables) by $\mathcal{L}_h(\mathcal{H})$ and the set of all density operators (quantum states) by $\mathfrak{S}(\mathcal{H})$ on the Hilbert space \mathcal{H} . The relation between the Wigner-Yanase skew information and the uncertainty relation was studied in [7]. Moreover the relation between the Wigner-Yanase-Dyson skew information and the uncertainty relation was studied in [4, 9]. In our previous paper [9], we defined a

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generalized skew information and then derived a kind of an uncertainty relation. In the section 2, we introduce a new generalized Wigner-Yanase skew information. On a generalization of the original Wigner-Yanase skew information, our generalization is different from the Wigner-Yanase-Dyson skew information and a generalized skew information defined in our previous paper [9]. Moreover we define a new quantity by our generalized Wigner-Yanase skew information and then we derive the trace inequality expressing a kind of the uncertainty relation.

2. TRACE INEQUALITIES ON A GENERALIZED WIGNER-YANASE SKEW INFORMATION

Firstly we review the relation between the Wigner-Yanase skew information and the uncertainty relation. In quantum mechanical system, the expectation value of an observable H in a quantum state ρ is expressed by $Tr[\rho H]$. It is natural that the variance for a quantum state ρ and an observable H is defined by $V_\rho(H) \equiv Tr[\rho(H - Tr[\rho H]I)^2] = Tr[\rho H^2] - Tr[\rho H]^2$. It is famous that we have the Heisenberg's uncertainty relation:

$$(2.1) \quad V_\rho(A)V_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2$$

for a quantum state ρ and two observables A and B . The further strong result was given by Schrödinger

$$V_\rho(A)V_\rho(B) - |Cov_\rho(A, B)|^2 \geq \frac{1}{4}|Tr[\rho[A, B]]|^2,$$

where the covariance is defined by $Cov_\rho(A, B) \equiv Tr[\rho(A - Tr[\rho A]I)(B - Tr[\rho B]I)]$. However, the uncertainty relation for the Wigner-Yanase skew information failed. (See [7, 4, 9].)

$$I_\rho(A)I_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2.$$

Recently, S.Luo introduced the quantity $U_\rho(H)$ representing a quantum uncertainty excluding the classical mixture:

$$(2.2) \quad U_\rho(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_\rho(H))^2},$$

then he derived the uncertainty relation on $U_\rho(H)$ in [6]:

$$(2.3) \quad U_\rho(A)U_\rho(B) \geq \frac{1}{4}|Tr[\rho[A, B]]|^2.$$

Note that we have the following relation

$$(2.4) \quad 0 \leq I_\rho(H) \leq U_\rho(H) \leq V_\rho(H).$$

The inequality (2.3) is a refinement of the inequality (2.1) in the sense of (2.4).

In this section, we study one-parameter extended inequality for the inequality (2.3).

Definition 2.1. For $0 \leq \alpha \leq 1$, a quantum state ρ and an observable H , we define the Wigner-Yanase-Dyson skew information

$$(2.5) \quad I_{\rho, \alpha}(H) \equiv \frac{1}{2}Tr[(i[\rho^\alpha, H_0])(i[\rho^{1-\alpha}, H_0])]$$

and we also define

$$J_{\rho, \alpha}(H) \equiv \frac{1}{2}Tr[\{\rho^\alpha, H_0\}\{\rho^{1-\alpha}, H_0\}],$$

where $H_0 \equiv H - \text{Tr}[\rho H]I$ and we denote the anti-commutator by $\{X, Y\} = XY + YX$.

Note that we have

$$\frac{1}{2}\text{Tr} [(i[\rho^\alpha, H_0]) (i[\rho^{1-\alpha}, H_0])] = \frac{1}{2}\text{Tr} [(i[\rho^\alpha, H]) (i[\rho^{1-\alpha}, H])]$$

but we have

$$\frac{1}{2}\text{Tr} [\{\rho^\alpha, H_0\} \{\rho^{1-\alpha}, H_0\}] \neq \frac{1}{2}\text{Tr} [\{\rho^\alpha, H\} \{\rho^{1-\alpha}, H\}].$$

Then we have the following inequalities:

$$(2.6) \quad I_{\rho, \alpha}(H) \leq I_\rho(H) \leq J_\rho(H) \leq J_{\rho, \alpha}(H),$$

since we have $\text{Tr}[\rho^{1/2} H \rho^{1/2} H] \leq \text{Tr}[\rho^\alpha H \rho^{1-\alpha} H]$. (See [1, 2] for example.) If we define

$$(2.7) \quad U_{\rho, \alpha}(H) \equiv \sqrt{V_\rho(H)^2 - (V_\rho(H) - I_{\rho, \alpha}(H))^2},$$

as a direct generalization of Eq.(2.2), then we have

$$(2.8) \quad 0 \leq I_{\rho, \alpha}(H) \leq U_{\rho, \alpha}(H) \leq U_\rho(H)$$

due to the first inequality of (2.6). We also have

$$(2.9) \quad U_{\rho, \alpha}(H) = \sqrt{I_{\rho, \alpha}(H) J_{\rho, \alpha}(H)}.$$

Remark 2.2. From the inequalities (2.4), (2.6) and (2.8), our situation is that we have

$$0 \leq I_{\rho, \alpha}(H) \leq I_\rho(H) \leq U_\rho(H)$$

and

$$0 \leq I_{\rho, \alpha}(H) \leq U_{\rho, \alpha}(H) \leq U_\rho(H).$$

Therefore our first concern is the ordering between $I_\rho(H)$ and $U_{\rho, \alpha}(H)$. However we have no ordering between them. Because we have the following examples. We set the density matrix ρ and the observable H such as

$$\rho = \begin{pmatrix} 0.6 & 0.48 \\ 0.48 & 0.4 \end{pmatrix}, H = \begin{pmatrix} 1.0 & 0.5 \\ 0.5 & 5.0 \end{pmatrix}.$$

If $\alpha = 0.1$, then $U_{\rho, \alpha}(H) - I_\rho(H)$ approximately takes -0.14736 . If $\alpha = 0.2$, then $U_{\rho, \alpha}(H) - I_\rho(H)$ approximately takes 0.4451 .

Conjecture 2.3. Our second concern is to show an uncertainty relation with respect to $U_{\rho, \alpha}(H)$ as a direct generalization of the inequality (2.3) such that

$$(2.10) \quad U_{\rho, \alpha}(X)U_{\rho, \alpha}(Y) \geq \frac{1}{4}|\text{Tr}[\rho[X, Y]]|^2$$

However we have not found the proof of the above inequality (2.10). In addition, we have not found any counter-examples of the inequality (2.10) yet.

In the present paper, we introduce a generalized Wigner-Yanase skew information which is a generalization of the Wigner-Yanase skew information defined in Eq.(1.1), but different from the Wigner-Yanase-Dyson skew information defined in Eq.(2.5).

Definition 2.4. For $0 \leq \alpha \leq 1$, a quantum state ρ and an observable H , we define a generalized Wigner-Yanase skew information by

$$K_{\rho,\alpha}(H) \equiv \frac{1}{2} \text{Tr} \left[\left(i \left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0 \right] \right)^2 \right]$$

and we also define

$$L_{\rho,\alpha}(H) \equiv \frac{1}{2} \text{Tr} \left[\left(\left\{ \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0 \right\} \right)^2 \right].$$

Remark 2.5. For two generalized Wigner-Yanase skew informations $I_{\rho,\alpha}(H)$ and $K_{\rho,\alpha}(H)$, we have the relation:

$$I_{\rho,\alpha}(H) \leq K_{\rho,\alpha}(H).$$

Indeed, for a spectral decomposition of ρ such as $\rho = \sum_k \lambda_k |\phi_k\rangle\langle\phi_k|$, we have the following expressions:

$$I_{\rho,\alpha}(H) = \frac{1}{2} \sum_{m,n} (\lambda_m^\alpha - \lambda_n^\alpha) (\lambda_m^{1-\alpha} - \lambda_n^{1-\alpha}) |\langle\phi_m|H|\phi_n\rangle|^2$$

and

$$K_{\rho,\alpha}(H) = \frac{1}{2} \sum_{m,n} \left(\frac{\lambda_m^\alpha - \lambda_n^\alpha + \lambda_m^{1-\alpha} - \lambda_n^{1-\alpha}}{2} \right)^2 |\langle\phi_m|H|\phi_n\rangle|^2.$$

By simple calculations, we see

$$\left(\frac{\lambda_m^\alpha - \lambda_n^\alpha + \lambda_m^{1-\alpha} - \lambda_n^{1-\alpha}}{2} \right)^2 - (\lambda_m^\alpha - \lambda_n^\alpha) (\lambda_m^{1-\alpha} - \lambda_n^{1-\alpha}) \geq 0.$$

Throughout this section, we put $X_0 \equiv X - \text{Tr}[\rho X]I$ and $Y_0 \equiv Y - \text{Tr}[\rho Y]I$. Then we show the following trace inequality.

Theorem 2.6. For a quantum state ρ and observables X, Y and $\alpha \in [0, 1]$, we have

$$(2.11) \quad W_{\rho,\alpha}(X) W_{\rho,\alpha}(Y) \geq \frac{1}{4} \left| \text{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2$$

where

$$W_{\rho,\alpha}(X) \equiv \sqrt{K_{\rho,\alpha}(X) L_{\rho,\alpha}(X)}.$$

Proof: Putting

$$(2.12) \quad M \equiv i \left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, X_0 \right] x + \left\{ \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, Y_0 \right\}$$

for any $x \in \mathbb{R}$, then we have

$$\begin{aligned} 0 &\leq \text{Tr}[M^* M] \\ &= \left(\frac{1}{4} \text{Tr} \left[(i[\rho^\alpha, X_0])^2 + (i[\rho^{1-\alpha}, X_0])^2 \right] + I_{\rho,\alpha}(X) \right) x^2 \\ &\quad + \frac{1}{2} \text{Tr} \left[(i[\rho^\alpha, X_0] + i[\rho^{1-\alpha}, X_0]) (\{\rho^\alpha, Y_0\} + \{\rho^{1-\alpha}, Y_0\}) \right] x \\ &\quad + \left(\frac{1}{4} \text{Tr} \left[\{\rho^\alpha, Y_0\}^2 + \{\rho^{1-\alpha}, Y_0\}^2 \right] + J_{\rho,\alpha}(Y) \right). \end{aligned}$$

Therefore we have

$$\begin{aligned} & \frac{1}{4} \left| \text{Tr} \left[(\rho^\alpha + \rho^{1-\alpha})^2 (i[X, Y]) \right] \right|^2 \\ & \leq 4 \left(\frac{1}{4} \text{Tr} \left[(i[\rho^\alpha, X_0])^2 + (i[\rho^{1-\alpha}, X_0])^2 \right] + I_{\rho, \alpha}(X) \right) \\ & \quad \times \left(\frac{1}{4} \text{Tr} \left[\{\rho^\alpha, Y_0\}^2 + \{\rho^{1-\alpha}, Y_0\}^2 \right] + J_{\rho, \alpha}(Y) \right), \end{aligned}$$

since we have

$$\text{Tr} \left[(i[\rho^\alpha, X_0] + i[\rho^{1-\alpha}, X_0]) (\{\rho^\alpha, Y_0\} + \{\rho^{1-\alpha}, Y_0\}) \right] = \text{Tr} \left[(\rho^\alpha + \rho^{1-\alpha})^2 (i[X, Y]) \right].$$

As similar as we have

$$\begin{aligned} & \frac{1}{4} \left| \text{Tr} \left[(\rho^\alpha + \rho^{1-\alpha})^2 (i[X, Y]) \right] \right|^2 \\ & \leq 4 \left(\frac{1}{4} \text{Tr} \left[(i[\rho^\alpha, Y_0])^2 + (i[\rho^{1-\alpha}, Y_0])^2 \right] + I_{\rho, \alpha}(Y) \right) \\ & \quad \times \left(\frac{1}{4} \text{Tr} \left[\{\rho^\alpha, X_0\}^2 + \{\rho^{1-\alpha}, X_0\}^2 \right] + J_{\rho, \alpha}(X) \right). \end{aligned}$$

By the above two inequalities, we have

$$W_{\rho, \alpha}(X) W_{\rho, \alpha}(Y) \geq \frac{1}{4} \left| \text{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2.$$

□

Corollary 2.7. For a quantum state ρ and observables (possibly unbounded operators) X, Y and $\alpha \in [0, 1]$, if we have the relation $[X, Y] = \frac{1}{2\pi i} I$ on $\mathbf{dom}(XY) \cap \mathbf{dom}(YX)$ and ρ is expressed by $\rho = \sum_k \lambda_k |\phi_k\rangle \langle \phi_k|$, $|\phi_k\rangle \in \mathbf{dom}(XY) \cap \mathbf{dom}(YX)$, then

$$W_{\rho, \alpha}(X) W_{\rho, \alpha}(Y) \geq \frac{1}{4} |\text{Tr} [\rho [X, Y]]|^2.$$

Proof: It follows from Theorem 2.6 and the following inequality:

$$\frac{1}{4} \left| \text{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2 \geq \frac{1}{4} |\text{Tr} [\rho [X, Y]]|^2,$$

whenever we have the canonical commutation relation such as $[X, Y] = \frac{1}{2\pi i} I$.

□

Remark 2.8. Theorem 2.6 is not trivial one in the sense of the following (i) and (ii).

- (i) Since the arithmetic mean is greater than the geometric mean, $\text{Tr} \left[(i[\rho^\alpha, X_0])^2 \right] \geq 0$ and $\text{Tr} \left[(i[\rho^{1-\alpha}, X_0])^2 \right] \geq 0$ imply $K_{\rho, \alpha}(X) \geq I_{\rho, \alpha}(X)$, by the use of Schwarz's inequality. Similarly, $\text{Tr} \left[\{\rho^\alpha, Y_0\}^2 \right] \geq 0$ and $\text{Tr} \left[\{\rho^{1-\alpha}, Y_0\}^2 \right] \geq 0$ imply $L_{\rho, \alpha}(Y) \geq J_{\rho, \alpha}(Y)$. We then have $W_{\rho, \alpha}(X) \geq U_{\rho, \alpha}(X)$.

From the inequality (2.8) and the above, our situation is that we have

$$U_{\rho, \alpha}(H) \leq U_\rho(H)$$

and

$$U_{\rho,\alpha}(H) \leq W_{\rho,\alpha}(H).$$

Our third concern is the ordering between $U_{\rho}(H)$ and $W_{\rho,\alpha}(H)$. However, we have no ordering between them. Because we have the following examples. We set

$$\rho = \begin{pmatrix} 0.8 & 0.0 \\ 0.0 & 0.2 \end{pmatrix}, H = \begin{pmatrix} 2.0 & 3.0 \\ 3.0 & 1.0 \end{pmatrix}.$$

If we take $\alpha = 0.8$, then $U_{\rho}(H) - W_{\rho,\alpha}(H)$ approximately takes -0.0241367 . If we take $\alpha = 0.9$, then $U_{\rho}(H) - W_{\rho,\alpha}(H)$ approximately takes 0.404141 . This example actually shows that there exists a triplet of α , ρ and H such that $W_{\rho,\alpha}(H) < V_{\rho}(H)$, since we have $U_{\rho}(H) \leq V_{\rho}(H)$ in general.

- (ii) We have no ordering between $\left| \text{Tr} \left[\left(\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2$ and $|\text{Tr}[\rho[X, Y]]|^2$, by the following examples. If we take

$$\rho = \frac{1}{7} \begin{pmatrix} 2 & 2i & 1 \\ -2i & 3 & -2i \\ 1 & 2i & 2 \end{pmatrix}, X = \begin{pmatrix} 3 & 3 & -i \\ 3 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} 1 & -i & 1-i \\ i & 1 & i \\ 1+i & -i & 3 \end{pmatrix},$$

then we have

$$\left| \text{Tr} \left[\left(\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2 \simeq 0.348097, |\text{Tr}[\rho[X, Y]]|^2 \simeq 0.326531.$$

If we take

$$\rho = \frac{1}{7} \begin{pmatrix} 2 & 2i & 1 \\ -2i & 3 & -2i \\ 1 & 2i & 2 \end{pmatrix}, X = \begin{pmatrix} 3 & 3 & -i \\ 3 & 1 & 0 \\ i & 0 & 1 \end{pmatrix}, Y = \begin{pmatrix} 1 & -i & 0 \\ i & 1 & i \\ 0 & -i & 3 \end{pmatrix},$$

then we have

$$\left| \text{Tr} \left[\left(\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2 \simeq 0.304377, |\text{Tr}[\rho[X, Y]]|^2 \simeq 0.326531.$$

Remark 2.9. (i) If we take $M = \rho^{1/2}X_0x + \rho^{1/2}Y_0$ for any $x \in \mathbb{R}$ presented in Eq.(2.12), we recover the Heisenberg uncertainty relation Eq.(2.1) shown in [3].

(ii) If we take $\alpha = \frac{1}{2}$, then we recover the inequality (2.3) presented in [6].

(iii) We have another inequalities which are different from the inequality (2.11), by taking different self-adjoint operators M appeared in the proof of Theorem 2.6.

Conjecture 2.10. Our fourth concern is whether the following inequality:

$$(2.13) \quad U_{\rho,\alpha}(X)U_{\rho,\alpha}(Y) \geq \frac{1}{4} \left| \text{Tr} \left[\left(\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2$$

holds or not. However we have not found its proof and any counter-examples yet.

$K_{\rho,\alpha}(H)$ and $L_{\rho,\alpha}(H)$ are respectively rewritten by

$$K_{\rho,\alpha}(H) = \text{Tr} \left[\left(\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right)^2 H_0^2 - \left(\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right) H_0 \left(\frac{\rho^{\alpha} + \rho^{1-\alpha}}{2} \right) H_0 \right]$$

and

$$L_{\rho,\alpha}(H) = \text{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 H_0^2 + \left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right) H_0 \left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right) H_0 \right].$$

Thus we have

$$\frac{1}{2} \text{Tr} \left[\left(i \left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0 \right] \right)^2 \right] = \frac{1}{2} \text{Tr} \left[\left(i \left[\frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H \right] \right)^2 \right]$$

but we have

$$\frac{1}{2} \text{Tr} \left[\left(\left\{ \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H_0 \right\} \right)^2 \right] \neq \frac{1}{2} \text{Tr} \left[\left(\left\{ \frac{\rho^\alpha + \rho^{1-\alpha}}{2}, H \right\} \right)^2 \right].$$

In addition, we have $L_{\rho,\alpha}(H) \geq K_{\rho,\alpha}(H)$ which implies

$$W_{\rho,\alpha}(H) \equiv \sqrt{K_{\rho,\alpha}(H)L_{\rho,\alpha}(H)} \geq \sqrt{K_{\rho,\alpha}(H)K_{\rho,\alpha}(H)} \geq K_{\rho,\alpha}(H).$$

Therefore our fifth concern is whether the following inequality for $\alpha \in [0, 1]$ holds or not:

$$(2.14) \quad K_{\rho,\alpha}(X)K_{\rho,\alpha}(Y) \geq \frac{1}{4} \left| \text{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2.$$

However this inequality fails, because we have a counter-example. If we set $\alpha = \frac{1}{2}$ and

$$\rho = \frac{1}{4} \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}, X = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then we have,

$$K_{\rho,\alpha}(X)K_{\rho,\alpha}(Y) = I_\rho(X)I_\rho(Y) = \left(\frac{1-\sqrt{3}}{2} \right)^2$$

and

$$\frac{1}{4} \left| \text{Tr} \left[\left(\frac{\rho^\alpha + \rho^{1-\alpha}}{2} \right)^2 [X, Y] \right] \right|^2 = \frac{1}{4} |\text{Tr} [\rho [X, Y]]|^2 = \frac{1}{4}.$$

Thus the inequality (2.14) does not hold in general.

Before closing this section, we reconsider the ordering $W_{\rho,\alpha}(H)$ and $V_\rho(H)$, although we have already stated an example of the triplet α, ρ and H satisfying $W_{\rho,\alpha}(H) < V_\rho(H)$ in the last line of (i) of Remark 2.8. If we set $\alpha = \frac{1}{5}$ and

$$\rho = \begin{pmatrix} 0.3 & 0.45 \\ 0.45 & 0.7 \end{pmatrix}, H = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

Then $V_\rho(H) - W_{\rho,\alpha}(H)$ approximately takes -0.3072 . If we set $\alpha = \frac{1}{5}$ and

$$\rho = \begin{pmatrix} 0.3 & 0.4 \\ 0.4 & 0.7 \end{pmatrix}, H = \begin{pmatrix} 1 & 3 \\ 3 & 1 \end{pmatrix}.$$

Then $V_\rho(H) - W_{\rho,\alpha}(H)$ approximately takes 0.682011 . Therefore we have no ordering between $W_{\rho,\alpha}(H)$ and $V_\rho(H)$. Thus it is natural for us to have an interest in the following conjecture, since we have $K_{\rho,\alpha}(H) \leq W_{\rho,\alpha}(H)$ in general.

Conjecture 2.11. Our final concern is whether the following inequality:

$$(2.15) \quad K_{\rho,\alpha}(H) \leq V_{\rho}(H), \quad \alpha \in [0, 1]$$

holds or not. However we have not found its proof and any counter-examples yet.

3. CONCLUDING REMARKS

As we have seen, we introduced a generalized Wigner-Yanase skew information $K_{\rho,\alpha}(H)$ and then defined a new quantity $W_{\rho,\alpha}(H)$. We note that our generalised Wigner-Yanase skew information $K_{\rho,\alpha}(H)$ is different type of the Wigner-Yanase-Dyson skew information $I_{\rho,\alpha}(H)$. For the quantity $K_{\rho,\alpha}(H)$, we do not have a trace inequality related to an uncertainty relation. However, we showed that we have a trace inequality related to an uncertainty relation for the quantity $W_{\rho,\alpha}(H)$. This inequality is a non-trivial one-parameter extension of the uncertainty relation Eq.(2.3) shown by S.Luo in [6]. In addition, we studied several trace inequaities on informational quantities.

Finally, we give another generalized trace inequality of the inequality (2.3). For a quantum state ρ an observable H and $\alpha \in [0, 1]$, we define

$$Z_{\rho,\alpha}(H) \equiv \frac{1}{4} \sqrt{\text{Tr} [(i[\rho^{\alpha}, H_0])^2] \text{Tr} [(i[\rho^{1-\alpha}, H_0])^2] \text{Tr} [\{\rho^{\alpha}, H_0\}^2] \text{Tr} [\{\rho^{1-\alpha}, H_0\}^2]},$$

with $H_0 \equiv H - \text{Tr}[\rho H]I$. Then we have the following inequality

$$(3.1) \quad \sqrt{Z_{\rho,\alpha}(X)Z_{\rho,\alpha}(Y)} \geq \frac{1}{4} \left| \text{Tr} [\rho^{2\alpha}[X, Y]] \text{Tr} [\rho^{2(1-\alpha)}[X, Y]] \right|,$$

for a quantum state ρ , two observables X, Y and $\alpha \in [0, 1]$. We note that the inequality (3.1) recovers the inequality (2.3) by taking $\alpha = 1/2$ and we do not have any weak-strong relation between the inequality (2.11) and the inequality (3.1).

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